MA20218: Analysis 2A

Problem Sheet 0: Sequences and series

Real sequences and series:

1. Calculate the $\liminf$ and $\limsup$ of the following sequences:
   - $\alpha_k = \frac{(-1)^k(k+5)}{k}$;
   - $\alpha_k = 5 + \frac{\sin k}{k}$.
2. Use the root test to determine whether the following real series converge:

$$\sum_{k=1}^{\infty} \frac{4^k}{3^{k+2}}, \quad \sum_{k=1}^{\infty} \frac{5^k}{3^k(k^4+2)}, \quad \sum_{k=1}^{\infty} \frac{k^k}{2^{k^2}}.$$

Sequences and series of functions:

a. Show that the sequence $(f_k)_k$, with $f_k : [0,1) \to \mathbb{R}$ defined by

$$f_k(x) = k^2 x^k, \qquad \text{for every } x \in [0,1),$$

converges pointwise to 0 as $k \to \infty$. What happens at $x = 1$?

b. Consider the sequence of functions $(f_k)_k$, with $f_k : \mathbb{R} \to \mathbb{R}$ defined by

$$f_k(x) = \frac{\sin(kx+3)}{\sqrt{k+1}}, \qquad \text{for every } x \in \mathbb{R}.$$

Show that it converges pointwise.

c. Prove that the sequence in a. does not converge uniformly in $[0,1)$.

d. Prove that the sequence in b. converges uniformly on $\mathbb{R}$.